AP Calculus

Chapter 5

Section 5-5

We say that differentiable curves are always **locally linear.**

In our study of the derivative we have frequently referred to the "tangent line to the curve" at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency.

$Y_1=(x^2+.0001)^\wedge(.25)+.9$

EXPLORATION 1 Appreciating Local Linearity

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence "locally linear" there. Let us explore the significance of this fact with the help of a graphing calculator.

1. Graph $y = f(x)$ in the "ZoomDecimal" window. What appears to be the behavior of the function at the point $(0, 1)$?
2. Show algebraically that f is differentiable at $x = 0$. What is the equation of the tangent line at $(0, 1)$?
3. Now zoom in repeatedly, keeping the cursor at $(0, 1)$. What is the long-range outcome of repeated zooming?
4. The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

$m=f'(a)$

$(a, f(a))$

DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of f at a.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a.** The point $x = a$ is the **center** of the approximation.

$$y-y_1=m(x-x_1)$$

$$y=y_1+m(x-x_1)$$

$$L(x)=f(a)+f'(a)(x-a)$$

Find the linearization of $f(x) = \sqrt[3]{1 - x}$ at x = 0, and use it to approximate $\sqrt[3]{.97}$. Use a calculator to determine the accuracy of the approximation.

$f'(x) = \frac{1}{3}(1-x)^{-2/3}(-1)$

$x = 0$

$f'(x) = -\frac{1}{3\sqrt[3]{(1-x)^2}}$

$f'(0) = -\frac{1}{3}$

$(0, f(0)) = (0,1)$

$\sqrt[3]{.97} = \sqrt[3]{1-.03}$

$L(x) = f(0) + f'(0)(x-0)$

$.97 = 1 - x$

$-1 \quad -1$

$-.03 = -x$

$.03 = x$

$L(x) = 1 - \frac{1}{3}x$

$L(.03) = 1 - \frac{1}{3}(.03)$

$L(.03) = 1 - .01 = .99$

$\sqrt[3]{.97} \approx .9898982992$

$.0001017$

Find the linearization of $f(x) = \sin^2 x - \frac{1}{4}$ at $x = \frac{\pi}{6}$, and use it to approximate $f(0.5)$. Use a calculator to determine the accuracy of the approximation.

$f(x) = (\sin x)^2 - 1/4$

$f'(x) = 2\sin x \cos x$

$f'\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$

$f'(\pi/6) = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$f'(\pi/6) = \frac{\sqrt{3}}{2}$

$(\pi/6, f(\pi/6)) \rightarrow (\pi/6, 0)$

$L(x) = 0 + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

Use linearization to approximate $\sqrt{66}$ and $\sqrt[3]{66}$.

$$L(x) = \overset{y_1}{f(a)} + \overset{m}{f'(a)}(x - \overset{x_1}{a})$$

$y = \sqrt{64+x}$ $x = a = 0$

$y' = \frac{1}{2}(64+x)^{-1/2}$

$f(a) = 8$

$f'(a) = \frac{1}{16}$

$L(x) = 8 + \frac{1}{16}(x-0)$ $64 + x = 66$

$L(2) = 8 + \frac{1}{16}(2-0)$ $x = 2$

$8\frac{1}{8} = 8.125$

$\sqrt{66} \approx 8.124038\ldots$

Use linearization to approximate $\sqrt{66}$ and $\sqrt[3]{66}$.

$y = \sqrt[3]{64+x}$ $x = 0$

$y' = \frac{1}{3}(64+x)^{-2/3}$

$y'(0) = \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}$

$y(0) = 4$

$L(x) = 4 + \frac{1}{48}(x-0)$

$L(2) = 4 + \frac{1}{48}(2-0)$

$4\frac{1}{24} \approx 4.041\overline{6}$

$\sqrt[3]{66} \approx 4.04124002\ldots$

DEFINITION Differentials

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

$f'(x) = \frac{dy}{dx}$ ← differential ← differential

$dy \rightarrow \Delta y \rightarrow$ change in y

For the function $f(x) = 7x^4 - 18x$, find the differential dy and evaluate dy when x = 1 and dx = 0.2

$f'(x) = 28x^3 - 18$

$f'(1) = 10$

$dy = f'(1) \cdot 0.2$

$dy = 10 \cdot 0.2$

$dy = 2$

Find the differential

$d(\sec(x^2))$

$y = \sec(x^2)$

$dy = f'(x)\,dx$ $\quad y' = \sec(x^2)\tan(x^2)(2x)$

$dy = \sec(x^2)\tan(x^2)2x\,dx$

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

a → constant

$$df = f'(a)\,dx.$$

dy

The radius of a sphere increases from 5cm to 5.2cm. Use differentials to estimate the increase in volume. Compare the estimate with the actual change to find the estimation error.

$V = \frac{4}{3}\pi r^3$ $dr = 5.2 - 5 = 0.2$

$V(r) = \frac{4}{3}\pi r^3 \rightarrow V'(r) = 4\pi r^2$

$dV = V'(r)\,dr$ $V'(5) = 100\pi$

$dV = 100\pi \cdot 0.2$

$dV = 20\pi \text{ cm}^3 \rightarrow$ Est.

Actual) $\frac{4}{3}\pi(5.2)^3 - \frac{4}{3}\pi(5)^3 =$

A sample of molecules is kept at constant temperature. The pressure and volume vary inversely (PV=k where k is a constant). Use the differentials to estimate the increase in volume when the pressure of the sample increases in atms from 0.7 to 0.77. Compare the estimate with the actual change to find the estimation error.

Homework

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1 - 19 odd, 23 - 26 all, 31 - 36 all, 41, 59, 60