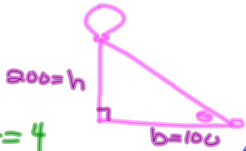
AP Calculus

Chapter 5

Section 5-6

A weather balloon moves vertically from level ground is rising at a constant rate of 4 feet per second. Its height is tracked from a station 100 feet from the launch point. How fast is the angle of elevation changing when the balloon is 200 feet in the air?



$\frac{dh}{dt} = 4$

$\frac{d\theta}{dt} = ?$

$\tan\theta = \frac{h}{b}$

$\tan\theta = \frac{h}{100}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{100} \cdot \frac{dh}{dt}$

$\tan\theta = \frac{200}{100}$

$\theta = \tan^{-1}(2)$

≈ 1.107

$\frac{\sec^2(1.107)\frac{d\theta}{dt}}{\sec^2(1.107)} = \frac{\frac{1}{100} \cdot 4}{\sec^2(1.107)}$

$\frac{d\theta}{dt} = .008$ radians/sec

Water is filling a right conical reservoir, point down. The reservoir's height is 21m and radius is 9m. To the nearest hundredth, how fast is the water level rising when the water is 5m deep and the rate of fill is 3 m³/min?

$\frac{dV}{dt} = 3$

$\frac{dh}{dt} = ?$

$\frac{h}{r} = \frac{21}{9} = \frac{7}{3}$

$\frac{h}{r} = \frac{7}{3}$

$\frac{3h}{7} = \cancel{\frac{7}{3}} r$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{\cancel{3}}\pi \frac{\cancel{9}^{3}}{49} h^2 \cdot h$

$V = \frac{3\pi}{49} h^3$

$\frac{dV}{dt} = \frac{3\pi}{49} 3h^2 \frac{dh}{dt}$

when $h=5$

$$\frac{\cancel{3}}{\frac{3\pi \cdot 75}{49}} = \frac{\frac{3\pi}{49} 3(5)^2 \frac{dh}{dt}}{\frac{3\pi \cdot 75}{49}}$$

$\frac{49}{75\pi} = \frac{dh}{dt} = \frac{49}{75\pi}$ m/min

Strategy for Related Rates

1. Identify the rates: one or more known, one unknown.
2. Model the problem, Ex: draw a diagram and label its parts.
3. Write an equation relating the variables
   - variables determined by known/unknown rates in part 1
4. Differentiate both sides of equation implicitly (w/ respect to time)
5. Substitute any known values for the variables and rates
6. Interpret the result as a rate (with correct units)

a) Write an equation relating the perimeter of a rectangle to its length and width.

$$P = 2L + 2W$$

b) How does dP/dt relate to dL/dt and dW/dt?

$$\frac{dP}{dt} = 2\frac{dL}{dt} + 2\frac{dW}{dt}$$

c) If the perimeter is increasing at a rate of 3 feet per minute and the length is increasing at a rate of 3 inches per minute, what is the rate at which the width is increasing?

$$\frac{dP}{dt} = 36 \text{ in/min}$$

$$\frac{dL}{dt} = 3 \text{ in/min}$$

$$36 = 2(3) + 2\frac{dW}{dt}$$

$$36 - 6 = 6 - 6 + 2\frac{dW}{dt}$$

$$30 = 2\frac{dW}{dt}$$

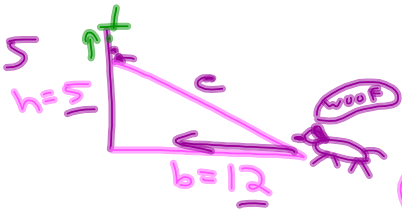
$$15 \text{ in/min} = \frac{dW}{dt}$$

Joe is walking his dog, Dog, when suddenly Dog takes off running at a squirrel sitting directly at the base of a telephone pole, which the squirrel starts climbing immediately. Dog runs 12.5 m/s and the squirrel at 9 m/s. At what rate is the actual distance between the animals changing when the squirrel is 5m high and Dog is 12m from the tree?

$$\frac{db}{dt} = -12.5$$

$$\frac{dh}{dt} = 9$$

$$\frac{dc}{dt} = ?$$



$$5^2 + 12^2 = c^2$$

$$25 + 144 = c^2$$

$$\sqrt{169} = \sqrt{c^2}$$

$$13 = c$$

$$h^2 + b^2 = c^2$$

$$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$\frac{\cancel{2}(5)(9) + \cancel{2}(12)(-12.5)}{\cancel{2}(13)} = \frac{2(13)\frac{dc}{dt}}{2 \cdot 13}$$

$$\frac{45 + (-150)}{13} = \frac{dc}{dt}$$

$$\frac{-105}{13} = \frac{dc}{dt}$$

$$-8.07 \text{ m/s} \approx \frac{dc}{dt}$$

The lateral surface of a cylindrical tank measuring 1.4 m from base to base is coated in a layer of ice of uniform thickness. The ice is melting at a constant rate of 300 cm³/min. What is the lateral surface area of the ice at the moment that it is decreasing at a rate of 0.7π cm²/min?

$\frac{dV}{dt}=300$

$\frac{dL}{dt}=-.7\pi$

$L=?$

$$\frac{L}{280\pi}=\frac{2\pi r(140)}{280\pi}$$

$$\frac{L}{280\pi}=r$$

$$V=\pi r^2 h$$

$$V=\pi\left(\frac{L}{280\pi}\right)^2(140)$$

$$\frac{dV}{dt}=\cancel{140\pi}(2)\left(\frac{L}{\cancel{280\pi}}\right)\left(\frac{1}{280\pi}\right)\frac{dL}{dt}$$

$$-300=\frac{L}{280\pi}(-.7\pi)$$

$$300=\frac{.7}{280}L$$

$$\frac{300\cdot\cancel{280}^{40}}{\cancel{.7}\,.1}=L$$

$$120000\text{ cm}^2=L$$

A spherical ballon is being inflated at a rate of 8 cm³/min. How quickly is the surface area of the ball increasing at the moment that the surface area is 900π cm²?

$\frac{dV}{dt}=8$

$\frac{dS}{dt}=?$

$S=900\pi$

$$V=\frac{4}{3}\pi r^3 \qquad S=4\pi r^2$$

$$\sqrt{\frac{S}{4\pi}}=r$$

$$V=\frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{3/2}$$

$$\frac{dV}{dt}=\frac{\cancel{4}^{2}}{\cancel{3}}\pi\cdot\frac{\cancel{3}}{\cancel{2}}\left(\frac{S}{4\pi}\right)^{1/2}\left(\frac{1}{4\pi}\right)\frac{dS}{dt}$$

$$8=\cancel{2}\cancel{\pi}\left(\frac{900\pi}{4\cancel{\pi}}\right)^{1/2}\left(\frac{1}{\cancel{4\pi}_{2}}\right)\frac{dS}{dt}$$

$$8=\frac{30}{2}\cdot\frac{1}{2}\frac{dS}{dt}$$

$$\frac{8\cdot4}{30}=\frac{dS}{dt}=\frac{16}{15}\text{ cm}^2/\text{min}$$

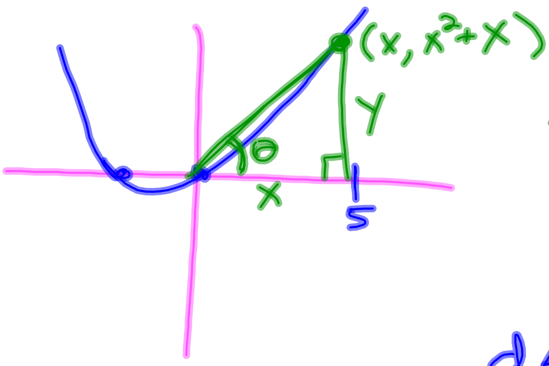
A particle is moving along the curve $y = x^2 + x$ and its horizontal position in changing at a constant rate of 6 units/minute. How is the angle of elevation, θ, changing when x = 5?

$(x+1)(x)$

$x = 5$

$\frac{dx}{dt} = 6$

$\frac{d\theta}{dt} = ?$



$\tan\theta = \frac{y}{x} = \frac{x^2+x}{x}$

$\tan\theta = x+1$

$\theta = \tan^{-1}(x+1)$

$\frac{d\theta}{dt} = \frac{1}{1+(x+1)^2} \cdot \frac{dx}{dt}$

$= \frac{1}{1+(6)^2} \cdot 6$

$\frac{d\theta}{dt} = \frac{6}{37}$ radians/min

A particle is moving along the curve $y = e^{-x+10}$ and its horizontal position in changing at a constant rate of 3 units/hr. To the nearest thousandth, how fast is the particle moving away from the origin when x=10?

Homework

Pages 255 - 257

3, 8, 9, 11, 13, 14, 16 - 20 all, 25, 27, 36 - 39 all